LINEAR STABILITY OF RING SYSTEMS

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ABSTRACT

We give a self-contained modern linear stability analysis of a system of n equal-mass bodies in circular orbit about a single more massive body. Starting with the mathematical description of the dynamics of the system, we form the linear approximation, compute all of the eigenvalues of the linear stability matrix, and finally derive inequalities that guarantee that none of these eigenvalues have a positive real part. In the end we rederive the result that Maxwell found for large n in his seminal paper on the nature and stability of Saturn's rings, which was published 150 years ago. In addition, we identify the exact matrix that defines the linearized system even when n is not large. This matrix is then investigated numerically (by computer) to find stability inequalities. Furthermore, using properties of circulant matrices, the eigenvalues of the large $4n \times 4n$ matrix are computed by solving n quartic equations, which further facilitates the investigation of stability. Finally, we implement an n-body simulator, and we verify that the threshold mass ratios that we derive mathematically or numerically do indeed identify the threshold between stability and instability. Throughout the paper we consider only the planar n-body problem so that the analysis can be carried out purely in complex notation, which makes the equations and derivations more compact, more elegant, and therefore, we hope, more transparent. The result is a fresh analysis that shows that these systems are always unstable for $2 \le n \le 6$, and for $n > 6$ they are stable provided that the central mass is massive enough. We give an explicit formula for this mass-ratio threshold.

Key word: planets: rings

1. INTRODUCTION

One hundred and fifty years ago, Maxwell (1859) was awarded the prestigious Adams Prize for a seminal paper on the stability of Saturn's rings. At that time, neither the structure nor the composition of the rings was known. Hence, Maxwell considered various scenarios such as the possibility that the rings were solid or liquid annuli or a myriad of small boulders. As a key part of this last possibility, Maxwell studied the case of n equal-mass bodies orbiting Saturn at a common radius and uniformly distributed about a circle of this radius. He concluded that, for large n, the ring ought to be stable provided that the following inequality is satisfied:

$$\text{mass(rings)} \le 2.298 \times \text{mass(Saturn)}/n^2.$$

The mathematical analysis that leads to this result has been scrutinized, validated, and generalized by a number of mathematicians over the years.

We summarize briefly some of the key historical developments. Tisserand (1889) derived the same stability criterion using an analysis in which he assumed that the ring has no effect on Saturn and that the highest vibration mode of the system controls stability. More recently, Willerding (1986) used the theory of density waves to show that Maxwell's results are correct in the limit as n goes to infinity. Pendse (1935) reformulated the stability problem so that it took into account the effect of the rings on the central body. He proved that for $n \le 6$ the system is unconditionally unstable. Inspired by this work, Salo & Yoder (1988) studied coorbital formations of n satellites for small values of n in which the satellites are not distributed uniformly around the central body. They showed that there are some stable asymmetric formations (such as the well-known case of a pair of ring bodies in L4/L5 position relative to

each other: i.e., one leading the other by $60°$). Scheeres & Vinh (1991) extended the analysis of Pendse to find the stability criterion as a function of the number of satellites when n is small. The resulting threshold depends on n, but for $n \ge 7$ it deviates only a small amount from the asymptotically derived value. More recently, Moeckel (1994) studied the linear stability of n-body systems for which the motion is given as a simple rotation about the center of mass and under the condition that all masses except one become vanishingly small. This latter condition greatly simplifies the analysis. Under these assumptions, Moeckel shows, using the invariant subspace of the linearized Hamiltonian, that symmetric ring systems are stable if and only if $n \ge 7$. For $n \le 6$ he gives some examples of the stable configurations discussed in Salo & Yoder (1988) in which the ring bodies are not uniformly distributed. Finally, Roberts (2000), expanding on Moeckel's work, obtained stability criteria that match those given by Scheeres & Vinh (1991).

In this paper we give a self-contained modern linear stability analysis of a system of equal-mass bodies in circular orbit about a single more massive body. We start with the mathematical description of the dynamics of the system. We then form the linear approximation, compute all of the eigenvalues of the matrix defining the linear approximation, and finally derive inequalities that guarantee that none of these eigenvalues have a positive real part. In the end we get exactly the same result that Maxwell found for large n. But, in addition, we identify the exact matrix that defines the linearized system even when n is not large. This matrix can then be investigated numerically to find stability inequalities even in cases in which n is not large. Furthermore, using properties of circulant matrices, the eigenvalues of the large $4n \times 4n$ matrix can be computed by solving n quartic equations, which

further facilitates the investigation. Finally, we implement an n-body simulator based on a leapfrog integrator (see Saha & Tremaine 1994; Hut et al. 1995), and we verify that the threshold mass ratios that we derive mathematically or numerically do indeed identify the threshold between stability and instability.

Throughout the paper we consider only the planar n-body problem. That is, we ignore any instabilities that might arise due to out-of-plane perturbations. Maxwell claimed, and others have confirmed, that these out-of-plane perturbations are less destabilizing than in-plane ones, and hence our analysis, while not fully general, does get to the right answer. Our main reason for wishing to restrict to the planar case is that we can then work in the complex plane and our entire analysis can be carried out purely in complex notation, which makes the equations and derivations more compact, more elegant, and therefore, we hope, more transparent.

Finally, we should point out that the relevance of this work to observed planetary rings is perhaps marginal, since real ring systems appear to be highly collisional and the energy dissipation associated with such collisions affects their stability in a fundamental way (see, e.g., Salo 1995).

2. EQUALLY SPACED, EQUAL-MASS BODIES IN A CIRCULAR RING ABOUT A MASSIVE BODY

Consider the multibody problem consisting of one large central body, for instance, Saturn, having mass M and n small bodies, such as boulders, each of mass m orbiting the large body in circular orbits uniformly spaced in a ring of radius r. Indices 0 to $n-1$ are used to denote the ring masses, and index n is used for Saturn. Throughout the paper we assume that $n \geq 2$. For the case $n = 1$, Lagrange proved that the system is stable for all mass ratios m/M.

The purpose of this section is to show that such a ring exists as a solution to Newton's law of gravitation. In particular, we derive the relationship between the angular velocity ω of the ring particles and their radius r from the central mass. We assume all bodies lie in a plane and therefore that complex variable notation is convenient. So, with $i = (-1)^{1/2}$ and $z = x + iy$, we can write the equilibrium solution for $j = 0, 1, \ldots, n-1$ as

$$z_j = r e^{i(\omega t + 2\pi j/n)}, \tag{1}$$

$$z_n = 0. \tag{2}$$

By symmetry (and exploiting our assumption that $n \geq 2$), force is balanced on Saturn itself. Now consider the ring bodies. Differentiating equation (1), we see that

$$\ddot{z}_j = -\omega^2 z_j. \tag{3}$$

From Newton's law of gravity we have that

$$\ddot{z}_j = -GM \frac{z_j - z_n}{|z_j - z_n|^3} + \sum_{k \neq j, n} Gm \frac{z_k - z_j}{|z_k - z_j|^3}. \tag{4}$$

Equations (3) and (4) allow us to determine ω, which is our first order of business. By symmetry it suffices to consider $j = 0$. It is easy to check that

$$z_k - z_0 = r e^{i\omega t} e^{\pi i k/n} 2i \sin(\pi k/n), \tag{5}$$

and hence that

$$|z_k - z_0| = 2r \sin(\pi k/n). \tag{6}$$

Substituting equations (5) and (6) into equation (4) and setting this equal to equation (3), we see that

$$-\omega^2 = -\frac{GM}{r^3} + \sum_{k=1}^{n-1} \frac{Gm}{4r^3} \frac{i e^{\pi i k/n}}{\sin^2(\pi k/n)} \tag{7}$$

$$= -\frac{GM}{r^3} - \frac{Gm}{4r^3} \sum_{k=1}^{n-1} \frac{1}{\sin(\pi k/n)} + i \frac{Gm}{4r^3} \sum_{k=1}^{n-1} \frac{\cos(\pi k/n)}{\sin^2(\pi k/n)}. \tag{8}$$

It is easy to check that the summation in the imaginary part on the right-hand side vanishes. Hence,

$$\omega^2 = \frac{GM}{r^3} + \frac{Gm}{r^3} I_n, \tag{9}$$

where

$$I_n = \frac{1}{4} \sum_{k=1}^{n-1} \frac{1}{\sin(\pi k/n)}. \tag{10}$$

With this choice of ω, the trajectories given by equations (1) and (2) satisfy Newton's law of gravitation.

3. FIRST-ORDER STABILITY

In order to carry out a stability analysis, we need to counterrotate the system so that all bodies remain at rest. We then perturb the system slightly and analyze the result.

A counterrotated system would be given by

$$e^{-i\omega t} z_j(t) = r e^{2\pi i j/n} = z_j(0).$$

In such a rotating frame of reference, each body remains fixed at its initial point. It turns out to be better to rotate the different bodies different amounts so that every ring body is repositioned to lie on the x-axis. In other words, for $j = 0, \ldots, n-1, n$ we define

$$w_j = u_j + iv_j = e^{-i(\omega t + 2\pi j/n)} z_j. \tag{11}$$

The advantage of repositioning every ring body to the positive real axis is that perturbations in the real part for any ring body represent radial perturbations, whereas perturbations in the imaginary part represent azimuthal perturbations. A simple counterrotation does not provide such a clear distinction between the two types of perturbations (and the associated stability matrix fails to have the circulant property that is crucial to all later analysis).

Differentiating equation (11) twice, we get

$$\ddot{w}_j = \omega^2 w_j - 2i\omega \dot{w}_j + e^{-i(\omega t + 2\pi j/n)} \ddot{z}_j. \tag{12}$$

From Newton's law of gravity, we see that

$$\ddot{w}_j = \omega^2 w_j - 2i\omega \dot{w}_j + \sum_{k \neq j} Gm_k \frac{\xi_{k,j}}{|\xi_{k,j}|^3}, \tag{13}$$

where

$$m_k = \begin{cases} m, & k = 0, 1, \ldots, n-1, \\ M, & k = n, \end{cases} \tag{14}$$

$$\xi_{k,j} = e^{i\theta_{k-j}} w_k - w_j, \tag{15}$$

$$\theta_k = 2\pi k/n. \tag{16}$$

Let $\delta w_j(t)$ denote variations about the fixed point given by

$$w_j \equiv \begin{cases} r, & j = 0, 1, \ldots, n-1, \\ 0, & j = n. \end{cases} \quad (17)$$

We compute a linear approximation to the differential equation describing the evolution of such a perturbation. Applying the quotient, chain, and product rules as needed, we get

$$\delta \ddot{w}_j = \omega^2 \, \delta w_j - 2i\omega \, \delta \dot{w}_j$$
$$+ \sum_{k \neq j} Gm_k \frac{|\xi_{k,j}|^3 \delta \xi_{k,j} - \xi_{k,j}(3/2)|\xi_{k,j}|(\xi_{k,j} \delta \bar{\xi}_{k,j} + \bar{\xi}_{k,j} \delta \xi_{k,j})}{|\xi_{k,j}|^6}$$
$$= \omega^2 \, \delta w_j - 2i\omega \, \delta \dot{w}_j - \frac{1}{2} \sum_{k \neq j} Gm_k \frac{|\xi_{k,j}|^2 \delta \xi_{k,j} + 3\xi_{k,j}^2 \delta \bar{\xi}_{k,j}}{|\xi_{k,j}|^5}, \quad (18)$$

where

$$\delta \xi_{k,j} = e^{i\theta_{k-j}} \delta w_k - \delta w_j, \quad \delta \bar{\xi}_{k,j} = e^{-i\theta_{k-j}} \delta \bar{w}_k - \delta \bar{w}_j.$$

The next step is to use equation (15) to re-express the $\xi_{k,j}$ values in terms of the w_k and w_j values, and then to substitute in the particular solution given by equation (17). Consider the case in which $j < n$. In this case we have

$$\xi_{k,j} = \begin{cases} r(e^{i\theta_{k-j}} - 1), & k < n, \\ -r, & k = n, \end{cases}$$

and therefore,

$$|\xi_{k,j}| = \begin{cases} 2r \sin(|\theta_{k-j}|/2), & k < n, \\ r, & k = n. \end{cases}$$

Substituting these into equation (18) and simplifying, we get

$$\delta \ddot{w}_j = \omega^2 \, \delta w_j - 2i\omega \, \delta \dot{w}_j$$
$$- \frac{GM}{2r^3} \left(e^{-i\theta_j} \delta w_n + 3e^{i\theta_j} \delta \bar{w}_n \right) + \frac{GM}{2r^3} \left(\delta w_j + 3\delta \bar{w}_j \right)$$
$$- \frac{Gm}{2r^3} \frac{1}{8} \sum_{k \neq j, n} \frac{e^{i\theta_{k-j}} \delta w_k - \delta w_j - 3e^{i\theta_{k-j}} \left(e^{-i\theta_{k-j}} \delta \bar{w}_k - \delta \bar{w}_j \right)}{\sin^3(|\theta_{k-j}|/2)}. \quad (19)$$

4. CHOICE OF COORDINATE SYSTEM

Without loss of generality, we can choose our coordinate system so that the center of mass remains fixed at the origin. Having done that, the perturbations δw_n and $\delta \bar{w}_n$ can be computed explicitly in terms of the other perturbations. Indeed, conservation of momentum implies that

$$m \sum_{k \neq n} \delta z_k + M \, \delta z_n = 0.$$

Hence,

$$\delta z_n = -\frac{m}{M} \sum_{k \neq n} \delta z_k.$$

From definition (11) of the w_k values in terms of the z_k values, it then follows that

$$e^{-i\theta_j} \, \delta w_n = -\frac{m}{M} \sum_{k \neq n} e^{i\theta_{k-j}} \, \delta w_k.$$

Making this substitution for $e^{-i\theta_j} \delta w_n$ and an analogous substitution for $e^{i\theta_j} \delta \bar{w}_n$ in equation (19), we see that

$$\delta \ddot{w}_j = \omega^2 \, \delta w_j - 2i\omega \, \delta \dot{w}_j$$
$$+ \frac{Gm}{2r^3} \sum_{k \neq n} \left(e^{i\theta_{k-j}} \delta w_k + 3e^{-i\theta_{k-j}} \delta \bar{w}_k \right) + \frac{GM}{2r^3} \left(\delta w_j + 3\delta \bar{w}_j \right)$$
$$- \frac{Gm}{2r^3} \frac{1}{8} \sum_{k \neq j, n} \frac{e^{i\theta_{k-j}} \delta w_k - \delta w_j - 3e^{i\theta_{k-j}} \left(e^{-i\theta_{k-j}} \delta \bar{w}_k - \delta \bar{w}_j \right)}{\sin^3(|\theta_{k-j}|/2)}. \quad (20)$$

5. CIRCULANT MATRIX

Switching to matrix notation, let W_j denote a shorthand for the column vector $[w_j \; \bar{w}_j]'$. In this notation we see that equation (20) together with its conjugates can be written as

$$\frac{d}{dt} \begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \\ \delta \dot{W}_0 \\ \delta \dot{W}_1 \\ \vdots \\ \delta \dot{W}_{n-1} \end{bmatrix} \approx \begin{bmatrix} & & & & I & & & \\ & & & & & I & & \\ & & & & & & \ddots & \\ & & & & & & & I \\ D & N_1 & \ldots & N_{n-1} & \Omega & & & \\ N_{n-1} & D & \ldots & N_{n-2} & & \Omega & & \\ \vdots & \vdots & & \vdots & & & \ddots & \\ N_1 & N_2 & \ldots & D & & & & \Omega \end{bmatrix} \begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \\ \delta \dot{W}_0 \\ \delta \dot{W}_1 \\ \vdots \\ \delta \dot{W}_{n-1} \end{bmatrix}, \quad (21)$$

where the D, Ω, and N_k notations are 2×2 complex matrices given by

$$D = \frac{3}{2}\omega^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{Gm}{2r^3} \begin{bmatrix} 1 - I_n + J_n/2 & 3 - 3J_n/2 \\ 3 - 3J_n/2 & 1 - I_n + J_n/2 \end{bmatrix},$$

$$N_k = \frac{Gm}{2r^3} \begin{bmatrix} e^{i\theta_k}(1 - J_{k,n}/2) & 3e^{-i\theta_k} + 3J_{k,n}/2 \\ 3e^{i\theta_k} + 3J_{k,n}/2 & e^{-i\theta_k}(1 - J_{k,n}/2) \end{bmatrix},$$

$$\Omega = 2i\omega \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

and where

$$I_{k,n} = \frac{1}{4\sin(\pi|k|/n)},$$

$$J_{k,n} = \frac{1}{4\sin^3(\pi|k|/n)},$$

and

$$I_n = \sum_{k=1}^{n-1} I_{k,n} \approx \frac{1}{2\pi} n \sum_{k=1}^{(n-1)/2} \frac{1}{k} \approx \frac{1}{2\pi} n \log \frac{n}{2}, \quad (22)$$

$$J_n = \sum_{k=1}^{n-1} J_{k,n} \approx \frac{1}{2\pi^3} n^3 \sum_{k=1}^{\infty} \frac{1}{k^3} = \frac{n^3}{2\pi^3} \zeta(3) = 0.01938 n^3. \quad (23)$$

TABLE 1
Estimates of the Stability Threshold

| | Threshold Value for γ | | | |
| | | | | |
n	Numerical	Eq. (43) (Even-n only)	Simulator	Roberts Results
2............................	0.000	*	[0.0, 0.007]	. . .
6............................	0.000	*	[0.0, 0.025]	. . .
7............................	2.452	. . .	[2.45, 2.46]	. . .
8............................	2.412	2.4121	[2.41, 2.42]	2.412
10...........................	2.375	2.3753	[2.37, 2.38]	2.375
12...........................	2.354	2.3543	[2.35, 2.36]	2.354
14...........................	2.341	2.3411	[2.34, 2.35]	2.341
20...........................	2.321	2.3213	[2.32, 2.33]	2.321
36...........................	2.306	2.3066	[2.30, 2.31]	. . .
50...........................	2.303	2.3031	[2.30, 2.31]	2.303
100..........................	2.300	2.2999	[2.30, 2.31]	2.300
101..........................	2.300	. . .	[2.30, 2.31]	2.300
500..........................	2.299	2.2987	[. . . , . . .]	2.299

Notes.—Stability thresholds are estimated as γ in an inequality of the type $m \leq \gamma M/n^3$. The second column contains numerically derived values obtained by a brute-force computation of the eigenvalues together with a simple binary search to find the first point at which an eigenvalue takes on a positive real part. The third column gives thresholds computed using eq. (43). The column of simulator values corresponds to results from running a leapfrog integrator and noting the smallest value of γ for which instability is clearly demonstrated. This is the larger of the pair of values shown. The smaller value is a nearby value for which the simulator was run 10 times longer with no overt indication of instability. Finally, the column of Roberts results shows values from Table 3 in Roberts (2000). Cells marked with an asterisk are cells for which an entry is not relevant, and cells marked with an ellipsis are cells for which an entry is unavailable.

Here the "approximate" symbol is used to indicate asymptotic agreement. That is, $a_n \approx b_n$ means that $a_n/b_n \to 1$ as $n \to \infty$, and $\zeta(3)$ denotes the value of the Riemann zeta function at 3. This constant is known as "Apéry's constant" (see, e.g., Arfken 1985).

Finally, note that in deriving equation (21) from equation (20) we have made repeated use of the identity

$$\sum_{k=1}^{n-1} \frac{e^{i\theta_k}}{\sin^3 |\theta_k|/2} = 4J_n - 8I_n. \tag{24}$$

Let A denote the matrix in equation (21). We need to find the eigenvalues of A and derive the necessary and sufficient conditions under which none of them have a positive real part. At this point we could resort to numerical computation to bracket a threshold for stability by doing a binary search to find the largest value of m/M for which none of the eigenvalues have a positive real part. We did such a search for some values of n. The results are shown in Table 1.

The eigenvalues are complex numbers for which there are nontrivial solutions to

$$\begin{bmatrix} & & & & I & & & & \\ & & & & & I & & & \\ & & & & & & \ddots & & \\ & & & & & & & I & \\ D & N_1 & \dots & N_{n-1} & \Omega & & & & \\ N_{n-1} & D & \dots & N_{n-2} & & \Omega & & & \\ \vdots & \vdots & & \vdots & & & \ddots & & \\ N_1 & N_2 & \dots & D & & & & \Omega \end{bmatrix} \begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \\ \delta \dot{W}_0 \\ \delta \dot{W}_1 \\ \vdots \\ \delta \dot{W}_{n-1} \end{bmatrix} = \lambda \begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \\ \delta \dot{W}_0 \\ \delta \dot{W}_1 \\ \vdots \\ \delta \dot{W}_{n-1} \end{bmatrix}. \tag{25}$$

The first n equations can be used to eliminate the "derivative" variables from the second set. That is,

$$\begin{bmatrix} \delta \dot{W}_0 \\ \delta \dot{W}_1 \\ \vdots \\ \delta \dot{W}_{n-1} \end{bmatrix} = \lambda \begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \end{bmatrix},$$

and therefore,

$$\begin{bmatrix} D & N_1 & \dots & N_{n-1} \\ N_{n-1} & D & \dots & N_{n-2} \\ \vdots & \vdots & & \vdots \\ N_1 & N_2 & \dots & D \end{bmatrix} \begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \end{bmatrix}$$
$$+ \lambda \begin{bmatrix} \Omega & & & \\ & \Omega & & \\ & & \ddots & \\ & & & \Omega \end{bmatrix} \begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \end{bmatrix} = \lambda^2 \begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \end{bmatrix}. \tag{26}$$

The matrix on the left-hand side is called a "block circulant matrix." Much is known about such matrices. In particular, it is easy to find the eigenvectors of such matrices. For general properties of block circulant matrices, see Tee (2005).

Let ρ denote an nth root of unity (i.e., $\rho = e^{2\pi ij/n}$ for some $j = 0, 1, \dots, n-1$), and let ξ be an arbitrary complex 2-vector. We look for solutions of the form

$$\begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \end{bmatrix} = \begin{bmatrix} \xi \\ \rho\xi \\ \vdots \\ \rho^{n-1}\xi \end{bmatrix}.$$

Substituting such a guess into equation (26), we see that each of the n rows reduces to one and the same thing:

$$\left(D + \rho N_1 + \ldots + \rho^{n-1} N_{n-1}\right)\xi + \lambda\Omega\xi = \lambda^2\xi.$$

There are nontrivial solutions to this 2×2 system if and only if

$$\det\left(D + \rho N_1 + \ldots + \rho^{n-1} N_{n-1} + \lambda\Omega - \lambda^2 I\right) = 0.$$

For each root of unity ρ, there are four values of λ that solve this equation (counting multiplicities). That makes a total of $4n$ eigenvalues and therefore provides all eigenvalues for the full system in equation (25).

6. EXPLICIT EXPRESSION FOR $\sum_{k=1}^{n-1} \rho^k N_k$

In order to compute the eigenvalues, it is essential that we compute $\sum_{k=1}^{n-1} \rho^k N_k$ as explicitly as possible. To this end, we note the following reduction and new definition:

$$\sum_{k=1}^{n-1} \rho^k J_{k,n} = \frac{1}{4}\sum_{k=1}^{n-1} \frac{e^{2\pi ijk/n}}{\sin^3(\theta_k/2)} = \frac{1}{4}\sum_{k=1}^{n-1} \frac{\cos(j\theta_k)}{\sin^3(\theta_k/2)} \equiv \tilde{J}_{j,n}.$$
(27)

Similarly,

$$\sum_{k=1}^{n-1} \rho^k e^{i\theta_k} J_{k,n} = \tilde{J}_{j+1,n},$$
(28)

$$\sum_{k=1}^{n-1} \rho^k e^{-i\theta_k} J_{k,n} = \tilde{J}_{j-1,n}.$$
(29)

We also compute

$$\sum_{k=1}^{n-1} \rho^k e^{i\theta_k} = \sum_{k=1}^{n-1} e^{ij\theta_k} e^{i\theta_k} = \sum_{k=1}^{n-1} e^{i(j+1)\theta_k}$$
$$= \begin{cases} n-1, & j = n-1, \\ -1, & \text{otherwise.} \end{cases}$$
(30)

$$\sum_{k=1}^{n-1} \rho^k e^{-i\theta_k} = \begin{cases} n-1, & j = 1, \\ -1, & \text{otherwise.} \end{cases}$$
(31)

Substituting the definition of N_k into $\sum_{k=1}^{n-1} \rho^k N_k$ and making use of equations (27)–(31), we get

$$\sum_{k=1}^{n-1} \rho^k N_k$$
$$= \frac{Gm}{2r^3}\begin{bmatrix} -1 + n\delta_{j=n-1} - \frac{1}{2}\tilde{J}_{j+1,n} & -3 + 3n\delta_{j=1} + \frac{3}{2}\tilde{J}_{j,n} \\ -3 + 3n\delta_{j=n-1} + \frac{3}{2}\tilde{J}_{j,n} & -1 + n\delta_{j=1} - \frac{1}{2}\tilde{J}_{j-1,n} \end{bmatrix},$$
(32)

where $\delta_{j=k}$ denotes the Kronecker delta (i.e., 1 when $j = k$ and 0 otherwise).

7. SOLVING $\det\left(D + \sum_{k=1}^{n-1} \rho^k N_k + \lambda\Omega - \lambda^2 I\right) = 0$

Assembling the results from the previous sections, we see that

$$D + \sum_{k=1}^{n-1} \rho^k N_k + \lambda\Omega - \lambda^2 I$$
$$= \begin{bmatrix} \frac{3}{2}\omega^2 + \frac{1}{2}\alpha_{j+1}^2 - \beta^2 - 2i\omega\lambda - \lambda^2 & \frac{3}{2}\omega^2 - \frac{3}{2}\alpha_j^2 \\ \frac{3}{2}\omega^2 - \frac{3}{2}\alpha_j^2 & \frac{3}{2}\omega^2 + \frac{1}{2}\alpha_{j-1}^2 - \beta^2 + 2i\omega\lambda - \lambda^2 \end{bmatrix}$$
$$+ \frac{Gm}{2r^3}\begin{bmatrix} n\delta_{j=n-1} & 3n\delta_{j=1} \\ 3n\delta_{j=n-1} & n\delta_{j=1} \end{bmatrix},$$
(33)

where α_j^2 and β^2 are shorthand for the expressions

$$\alpha_j^2 = \frac{Gm}{2r^3}\left(J_n - \tilde{J}_{j,n}\right) \geq 0,$$

$$\beta^2 = \frac{Gm}{2r^3} I_n \geq 0,$$

and, as a reminder, I_n and J_n are defined by equations (22) and (23), respectively, whereas $\tilde{J}_{j,n}$ is defined by equation (27).

It turns out in our subsequent analysis that the root of unity given by $j = n/2$ is the most critical one for stability, at least for $n \geq 7$. For $n = 2, \ldots, 6$ the instability stems from the eigenvectors associated with $j = 1$ and $j = n - 1$. We analyze the key cases. But first we note that the critical $j = n/2$ case corresponds to perturbations in which every other body is perturbed in the opposite direction. And, more importantly, it does not matter what the direction of the perturbation is. That is, if body 0 is advanced azimuthally, then all of the even-numbered bodies are advanced azimuthally, and all of the odd-numbered bodies are retarded by the same amount. Similarly, if body 0 is pushed outward radially, then all of the even-numbered bodies are also pushed outward, whereas the odd-numbered bodies are pulled inward. Azimuthal and radial perturbations contribute equally to instability.

7.1. The Case in Which n Is Arbitrary and j Is neither 1 nor n − 1

Assuming that j is neither 1 nor $n - 1$, we see that

$$\det\left(D + \sum_{k=1}^{n-1} \rho^k N_k + \lambda\Omega - \lambda^2 I\right)$$
$$= \left(\frac{3}{2}\omega^2 + \frac{1}{2}\alpha_{j+1}^2 - \beta^2 - 2i\omega\lambda - \lambda^2\right)$$
$$\times \left(\frac{3}{2}\omega^2 + \frac{1}{2}\alpha_{j-1}^2 - \beta^2 + 2i\omega\lambda - \lambda^2\right)$$
$$- \left(\frac{3}{2}\omega^2 - \frac{3}{2}\alpha_j^2\right)^2.$$
(34)

Expanding out the products on the right-hand side in equation (34), we get that

$$\det\left(D + \sum_{k=1}^{n-1} \rho^k N_k + \lambda\Omega - \lambda^2 I\right) = \lambda^4 + A_j\lambda^2 + iB_j\lambda + C_j = 0,$$
(35)

where

$$A_j = \omega^2 - \frac{1}{2}\left(\alpha_{j-1}^2 + \alpha_{j+1}^2\right) + 2\beta^2,$$
(36)

$$B_j = -\omega\left(\alpha_{j-1}^2 - \alpha_{j+1}^2\right), \tag{37}$$

$$C_j = 3\omega^2\left[\frac{1}{4}\left(\alpha_{j-1}^2 + \alpha_{j+1}^2\right) + \frac{3}{2}\alpha_j^2 - \beta^2\right]$$
$$+ \left[\frac{1}{4}\left(\alpha_{j-1}^2 + \alpha_{j+1}^2\right) - \beta^2\right]^2$$
$$- \frac{1}{16}\left(\alpha_{j-1}^2 - \alpha_{j+1}^2\right)^2 - \frac{9}{4}\alpha_j^4. \tag{38}$$

7.1.1. *The Subcase in Which n Is Even and j = n/2*

In the subcase in which n is even and $j = n/2$, it is easy to see by symmetry that $\alpha_{n/2+1} = \alpha_{n/2-1}$. To emphasize the equality, we denote this common value by $\alpha_{n/2\pm1}$. Equations (36)–(38) simplify significantly. The result is

$$\det\left(D + \sum_{k=1}^{n-1}\rho^k N_k + \lambda\Omega - \lambda^2 I\right)$$
$$= \lambda^4 + \left(\omega^2 - \alpha_{n/2\pm1}^2 + 2\beta^2\right)\lambda^2$$
$$+ 3\omega^2\left(\frac{1}{2}\alpha_{n/2\pm1} + \frac{3}{2}\alpha_{n/2}^2 - \beta^2\right)$$
$$+ \left(\frac{1}{2}\alpha_{n/2\pm1}^2 - \beta^2\right)^2 - \frac{9}{4}\alpha_{n/2}^4. \tag{39}$$

For a moment let us write this biquadratic polynomial (in λ) in a simple generic form and equate it to 0:

$$\lambda^4 + A\lambda^2 + C = 0.$$

The quadratic formula then tells us that

$$\lambda^2 = \frac{-A \pm \sqrt{A^2 - 4C}}{2}.$$

To get the eigenvalues we need to take square roots one more time. The only way for the resulting eigenvalues not to have a positive real part is for λ^2 to be real and nonpositive. Necessary and sufficient conditions for this are that

$$A \geq 0, \tag{40}$$
$$C \geq 0, \tag{41}$$
$$A^2 - 4C \geq 0. \tag{42}$$

It turns out that the third condition implies the first two (we leave verification of this fact to the reader). In terms of computable quantities, this third condition can be written, after simplification, as

$$\omega^4 + \left(-8\alpha_{n/2\pm1}^2 - 18\alpha_{n/2}^2 + 16\beta^2\right)\omega^2 + 9\alpha_{n/2}^4 \geq 0.$$

Again we use the quadratic formula to find that

$$\omega^2 \geq 4\alpha_{n/2\pm1}^2 + 9\alpha_{n/2}^2 - 8\beta^2$$
$$+ \sqrt{\left(4\alpha_{n/2\pm1}^2 + 9\alpha_{n/2}^2 - 8\beta^2\right)^2 - 9\alpha_{n/2}^4},$$

or

$$\omega^2 \leq 4\alpha_{n/2\pm1}^2 + 9\alpha_{n/2}^2 - 8\beta^2$$
$$- \sqrt{\left(4\alpha_{n/2\pm1}^2 + 9\alpha_{n/2}^2 - 8\beta^2\right)^2 - 9\alpha_{n/2}^4}.$$

It is the greater-than constraint that is relevant, and so we take the positive root. Finally, we recall that

$$\omega^2 = \frac{GM}{r^3} + \frac{Gm}{r^3}I_n, \quad \alpha_{n/2}^2 = \frac{Gm}{2r^3}\left(J_n - \tilde{J}_{n/2,n}\right),$$
$$\beta^2 = \frac{Gm}{2r^3}I_n,$$

and so the inequality on ω^2 reduces to

$$\frac{M}{m} \geq 2\left(J_n - \tilde{J}_{n/2\pm1,n}\right) + \frac{9}{2}\left(J_n - \tilde{J}_{n/2,n}\right) - 5I_n$$
$$+ \left\{\left[2\left(J_n - \tilde{J}_{n/2\pm1,n}\right) + \frac{9}{2}\left(J_n - \tilde{J}_{n/2,n}\right) - 4I_n\right]^2\right.$$
$$\left. - \frac{9}{4}\left(J_n - \tilde{J}_{n/2,n}\right)^2\right\}^{1/2}. \tag{43}$$

The third column in Table 1 shows thresholds computed using this inequality. It is clear that for even values of n greater than 7, this threshold matches the numerically derived threshold shown in the second column in the table. This suggests that inequalities analogous to equation (43) derived for $j \neq n/2$ are less restrictive than equation (43). The proof of this statement is obviously more complicated than the $j = n/2$ case because the general case includes a linear term ($B_j \neq 0$) that vanishes in the $j = n/2$ case. The linear term makes it impossible simply to use the quadratic formula, and therefore, any analysis involves a more general analysis of a quartic equation. Scheeres & Vinh (1991) analyzed the general case. Although their notations are different, the fundamental quantities are the same, and so their analysis is valid here as well. Rather than repeating their complete analysis, we simply outline the basic steps in § 7.1.2.

7.1.2. *The Subcase in Which j ≠ n/2*

Let $\alpha_{j\pm1}$ denote the average of α_{j+1} and α_{j-1}:

$$\alpha_{j\pm1} = (\alpha_{j+1} + \alpha_{j-1})/2.$$

If we were to assume for the moment, incorrectly, that terms involving the difference $\alpha_{j+1} - \alpha_{j-1}$ were not present in equations (36)–(38), then an analysis analogous to that given in § 7.1.1 would give us the inequality

$$\omega^2 \geq 4\alpha_{j\pm1}^2 + 9\alpha_j^2 - 8\beta^2 + \sqrt{\left(4\alpha_{j\pm1}^2 + 9\alpha_j^2 - 8\beta^2\right)^2 - 9\alpha_j^4}.$$

Next one uses the fact that $\alpha_{j\pm1}^2$ is unimodal as a function of j, taking its maximum value at $j = n/2$. Hence, the inequality associated with $j = n/2$ is the strictest of these inequalities. Finally, the difference terms are treated as small perturbations to this simple case, and a homotopy analysis shows that the $j = n/2$ case remains the strictest case even as the difference terms are fed in.

7.1.3. *Large n*

When n is large, $\tilde{J}_{n/2\pm1,n} \approx \tilde{J}_{n/2,n}$ and $J_n \gg I_n$. Furthermore,

$$
\tilde{J}_{n/2,n} \approx \frac{1}{2}\sum_{k=1}^{n/2}\frac{(-1)^k}{\sin^3(k\pi/n)} \approx \frac{n^3}{2\pi^3}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3}
$$

$$
= -\frac{3}{4}\frac{n^3}{2\pi^3}\sum_{k=1}^{\infty}\frac{1}{k^3} \approx -\frac{3}{4}\frac{1}{2}\sum_{k=1}^{n/2}\frac{1}{\sin^3(k\pi/n)} \approx -\frac{3}{4}J_n.
$$

Hence, equation (43) reduces to

$$
\frac{M}{m} \geq \frac{7}{8}\left(13 + 4\sqrt{10}\right)J_n,
$$

or, equivalently,

$$
m \leq \frac{M}{(7/8)\left(13 + 4\sqrt{10}\right)J_n} \approx \frac{2.299M}{n^3}, \tag{44}
$$

which is precisely the answer Maxwell obtained 150 years ago. Of course, we have assumed here that n is even. For the odd case, as $n \to \infty$, $|\alpha_{j-1} - \alpha_{j+1}| \to 0$, so that the odd quartic equation for $j = (n-1)/2$ reduces to the even equation for $j = n/2$, giving the same stability criteria as the even-particle case. This can be seen in our simulations as well. The cases of $n = 100$ and 101 give the same threshold to several significant figures.

7.2. *The Case in Which $j = 1$ or $j = n - 1$*

By symmetry, these two cases are the same. Hence, we consider only $j = 1$. Again after some manipulation in which we exploit the fact that $\alpha_0^2 = 0$, we arrive at

$$
\det\left(D + \sum_{k=1}^{n-1}\rho^k N_k + \lambda\Omega - \lambda^2 I\right) = \lambda^4 + A_j\lambda^2 + iB_j\lambda + C_j = 0, \tag{45}
$$

where

$$
A_1 = \omega^2 - \frac{1}{2}\left(n\gamma + \alpha_2^2\right) + 2\beta^2, \tag{46}
$$

$$
B_1 = -\omega\left(n\gamma - \alpha_2^2\right), \tag{47}
$$

$$
C_1 = 3\omega^2\left(\frac{1}{4}\alpha_2^2 + \frac{3}{2}\alpha_1^2 - \beta^2 - \frac{1}{2}n\gamma\right) + \left[\frac{1}{4}(n\gamma + \alpha_2^2) - \beta^2\right]^2
$$
$$
- \frac{1}{16}\left(n\gamma - \alpha_2^2\right)^2 - \frac{9}{4}\alpha_1^2\left(\alpha_1^2 - n\gamma\right), \tag{48}
$$

$$
\gamma = \frac{Gm}{r^3}. \tag{49}
$$

Note that the coefficient B_1 is imaginary, whereas the other three coefficients are real. This suggests making the substitution $\mu = i\lambda$. In terms of μ, equation (45) becomes a quartic equation with all real coefficients:

$$
\mu^4 - A_1\mu^2 + B_1\mu + C_1 = 0. \tag{50}
$$

This equation has at most four real roots. If it does have four real roots, then the corresponding values for λ are purely imaginary and the system could be stable. If, on the other hand, there are two or fewer real roots, then at least one pair of roots to equa-

tion (50) form a conjugate pair, and therefore, the corresponding pair of values for λ are such that one has a positive real part and the other a negative real part. Hence, in that case the system is demonstrably unstable. Simple numerical investigation reveals that this is precisely what happens when $2 \leq n \leq 6$ regardless of the mass ratio M/m.

To see why, let us consider just the case in which M/m is very large, and hence the ratio

$$
r = m/M
$$

is very close to 0. In this asymptotic regime,

$$
A_1 = aM, \quad B_1 = b\sqrt{M}m, \quad C_1 = cMm,
$$

where $a > 0$ and the sign of c is the same as the sign of $(1/4)\alpha_2^2 + (3/2)\alpha_1^2 - \beta^2 - (1/2)n\gamma$. Substituting rM for m and making the change of variables defined by $\nu = \mu/\sqrt{M}$, we get

$$
\nu^4 - a\nu^2 + br\nu + cr = 0.
$$

For $r = 0$ this equation reduces to $\nu^4 - a\nu^2 = 0$, which has three real roots: a positive one, a negative one, and a root of multiplicity 2 at $\nu = 0$. By continuity, for r small but nonzero the quartic still has a positive root and a negative root, but the double root at $\nu = 0$ can either disappear or split into a pair of real roots. Since this bifurcation takes place in the neighborhood of the origin, the quartic term can be ignored, and the equation in the neighborhood of zero reduces to a quadratic equation:

$$
-a\nu^2 + br\nu + cr = 0.
$$

This equation has two real roots if and only if its discriminant is nonnegative:

$$
r\left(rb^2 + 4ac\right) \geq 0.
$$

Hence, if c is negative there will not be a full set of real roots for r very small, and hence, the ring system will be unstable in that case. In other words, the system will be unstable if

$$
\frac{1}{4}\alpha_2^2 + \frac{3}{2}\alpha_1^2 - \beta^2 - \frac{1}{2}n\gamma < 0.
$$

This equation reduces to

$$
\sum_{k=1}^{n-1}\frac{1}{\sin(\pi k/n)} - n - \frac{1}{2}\cot\left(\frac{\pi}{2n}\right) < 0. \tag{51}
$$

It is easy to check that the expression is negative for $n = 2, \ldots,$ 6 and positive for $n \geq 7$. Therefore, we have proved that ring systems are unstable for $n = 2, \ldots,$ 6, at least when m is very small relative to M. We have not proved the result for larger values of m, but it seems that such a case should be even more unstable, which is certainly verified by our simulator.

8. COMPARISON WITH PRIOR RESULTS

Of the prior work summarized in § 1, the work of Roberts (2000) is the most recent, most rigorous, and most self-contained. In this section we compare our results with his. Like us, Roberts shows that the ring configuration is linearly unstable when $n \leq 6$. Furthermore, for $n \geq 7$ Roberts shows that the ring is linearly

stable if and only if $M > h_n m$, where h_n is a bifurcation value in a certain quartic polynomial. For n even, the formula he derives for h_n is

$$h_n = A + \sqrt{A^2 - B},$$

where

$$A = 2\sigma + 8P + 18Q, \quad B = 36Q^2 - 8\sigma P,$$

and

$$\sigma = \frac{1}{2} \sum_{k=1}^{n-1} \frac{1}{\sin \theta_k / 2} = 2I_n,$$

$$P = \sum_{k=1}^{n-1} \frac{1 - (-1)^k \cos \theta_k}{16 \sin^3 \theta_k / 2} = \frac{1}{4} \left(J_n - \tilde{J}_{n/2 \pm 1, n} \right),$$

$$Q = \sum_{k=1}^{n-1} \frac{\cos \theta_k - (-1)^k}{16 \sin^3 \theta_k / 2} = \frac{1}{4} \left(J_n - 2I_n - \tilde{J}_{n/2, n} \right).$$

In the above expressions we have used formulae (22), (23), (24), (27), and (28) to establish the alternate form of these expressions. From these alternate forms it is easy to see that the right-hand side in equation (43) is precisely h_n.

Furthermore, the fact that the asymptotic expression given by Roberts matches ours is easy to verify directly. Roberts wrote the asymptotic expression as

$$h_n \approx \frac{13 + 4\sqrt{10}}{2\pi^3} \sum_{1}^{\infty} \frac{1}{(2k-1)^2}.$$

To verify the agreement, it suffices to note that

$$\sum_{k=1}^{\infty} \frac{1}{k^3} = \sum_{k=1}^{\infty} \frac{1}{(2k)^3} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^3} = \frac{1}{8} \sum_{k=1}^{\infty} \frac{1}{k^3} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^3},$$

and therefore,

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^3} = \frac{7}{8} \sum_{k=1}^{\infty} \frac{1}{k^3}.$$

With this identity, it is easy to see that his expression matches ours given in equation (44).

9. RING DENSITY

Suppose that the linear density of the boulders is λ. That is, λ is the ratio of the diameter of one boulder to the separation between the centers of two adjacent boulders. Then the diameter of a single boulder is $\lambda(2\pi r/n)$. Hence, the volume of a single boulder is $(4\pi/3)(\lambda \pi r/n)^3$. Let δ denote the density of a boulder. Then the mass of a single boulder is $(4\pi/3)(\lambda \pi r/n)^3 \delta$. If we assume that the density of Earth is about 8 times that of a boulder (Earth's density is 5.5 and Saturn's moons have a density of about 0.7, being composed of porous water ice), then we have

$$\delta = \frac{1}{8} \frac{m_E}{(4\pi/3) r_E^3},$$

where m_E denotes the mass of Earth and r_E denotes its radius. Combining all of these factors and assuming the central mass is

equal to Saturn's mass and the ring's radius is about the radius of the Cassini division (120,000 km), we see that the upper bound on the linear density of boulders is

$$\lambda \leq \left(8 \frac{M/m_E}{25.65 \times 0.01938} \right)^{1/3} \frac{r_E}{r} = 0.219.$$

In other words, the linear density cannot exceed 22%; otherwise, the ring will be unstable. Of course, this is for a one-dimensional circular ring of ice boulders. Analysis of a two-dimensional annulus or the full three-dimensional case is naturally more complicated. Nonetheless, the 22% linear density figure matches surprisingly well with the measured optical density, which hovers around 0.05–2.5.

10. NUMERICAL RESULTS

We have computed stability thresholds three different ways for various finite n. First we numerically solved for all eigenvalues of the $4 \times 4n$ matrix in equation (25) and did a binary search to locate the smallest mass ratio M/m for which no eigenvalue has a positive real part. We then translated this threshold into a value of γ, expressed as

$$m \leq \gamma M / n^3,$$

and tabulated those results in the "Numerical" column in Table 1.

Second, for even values of n we used equation (43) to derive γ threshold values. These values are reported in the third column in Table 1. Note that for even values of n larger than 7, these results agree with those obtained numerically.

Finally, and perhaps most interestingly, the fourth column in the table are stability thresholds that were estimated using a simulator based on a leapfrog integrator (Saha & Tremaine 1994; Hut et al. 1995). In this column two values are given. For the larger value, instability has been decisively observed. However, verifying stability is more challenging since one should in principle run the simulator forever. Rather than waiting that long, we use the rule of thumb that if the system appears intact for a period of time 10 times greater than the time it took to demonstrate instability, then we deem the system stable at that mass. This is how the lower bounds in the table were obtained. It is our belief that a symplectic simulator provides the most convincing method of discriminating between stable and unstable orbits, at least when the number of bodies remains relatively small, for instance, up to a few hundred. Unstable orbits reveal themselves quickly, as the initial inaccuracy of double-precision arithmetic quickly cascades into something dramatic if the system is unstable. If, on the other hand, the system is stable, then the initial imprecisions simply result in an orbit that is close to but not identical to the intended orbit. The situation does not decay. Any reader who has never experimented with a good symplectic integrator is strongly encouraged to experiment with the Java applet posted online,[1] as hands-on experience can be very convincing.

Of course, the amazing thing about the simulator results is that they match the numerical results in the second column. The thresholds determined by linear stability analysis only tell us definitively that for m larger than the threshold, the system is necessarily unstable. But for m smaller than the threshold, the mathematical/numerical analysis says nothing, since in those

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[1] See http://www.princeton.edu/~rvdb/JAVA/astro/galaxy/StableRings.html.

cases the eigenvalues are all purely imaginary. Yet simulation confirms that the thresholds we have derived are truly necessary and sufficient conditions for stability.

As shown in § 7.2 for $2 \leq n \leq 6$, the system is unstable. The simulator verifies this. In these cases there is lots of room to roam before one body catches up to another. Even the tiniest masses

are unstable. The case $n = 2$ is especially interesting. For this case the two small bodies are at opposite sides of a common orbit. Essentially they are in the L3 position with respect to each other. For the restricted three-body problem (in which one body has zero mass) it is well known that L3 is unstable no matter what the mass ratio. Our results bear this out.

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